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Marginal Distributions of Wigner Function in a Mesoscopic L-C Circuit at Finite Temperature and Thermal Wigner Operator

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Wigner function in phase space has its physical meaning as marginal probability distribution in coordinate space and momentum space respectively, here we endow the Wigner function with a new physical meaning, i.e., its marginal distributions' statistical average for $q^2/(2C)$ and $p^2/(2L)$ are the energy stored in capacity and in inductance of a mesoscopic L-C circuit at finite temperature, respectively.

KEY WORDS: L-C circuit; thermal Wigner operator; IWOP technique; marginal distribution.

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1. INTRODUCTION

With the development of nanotechnology and microelectronics, it becomes more and more important to study the quantum characteristic of mesoscopic systems. A single L-C nondissipative circuit is a fundamental cell in electric circuits. Its quantization was first discussed by Louisell (1973) and some more progress has been made in this field in recent years (Fan *et al.*, 2000; Wang *et al.*, 2000; Song, 2003). Recall the Louisell's scheme, in which charge and current were quantized as the coordinate and momentum operators, respectively.

In this paper, we shall reveal that in the quantization theory of a mesoscopic L-C circuit at finite temperature, the Wigner function has a new physical meaning, i.e., its marginal distributions' statistical average for $q^2/(2C)$ and $p^2/(2L)$ are the

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temperature-related energy stored in capacity and in inductance of a mesoscopic L-C circuit at finite temperature, respectively. This new view endows the Wigner function with a more realistic physical interpretation. In Sec. 2 we shall discuss how the Umezawa-Takahashi thermal field dynamics (TFD) plays the role in quantizing a mesoscopic L-C circuit at finite temperature. Then in Sec. 3 based on the coherent thermal state representation (Fan *et al.*, 1998), we introduce the thermal Wigner operator. In Sec. 4 we calculate the Wigner function of thermal vacuum state. In Sec. 5 we calculate marginal distribution of the Wigner function of the thermal vacuum state.

2. THERMAL VACUUM STATE OF L-C CIRCUIT AT FINITE TEMPERATURE

Umezawa invented TFD to convert the ensemble average

$$\langle A\rangle = Z^{-1}(\beta)\mathrm{Tr}(Ae^{-\beta H}), \tag{1}$$

into the pure state average (Takahashi *et al.*, 1975)

$$\langle A\rangle = \langle 0(\beta)|\, A\, |0(\beta)\rangle\,. \tag{2}$$

The expense of this convenience is doubling the Hilbert space $|0\rangle \rightarrow \left|0, \tilde{0}\right\rangle$ and introducing the tilde operator $\tilde{a}^{+}$. For a harmonic oscillator system,

$$|0(\beta)\rangle = \exp[a^{+}\tilde{a}^{+}\tanh\theta]|0, \tilde{0}\rangle = S(\theta)|0, \tilde{0}\rangle, \tag{3}$$

where

$$S(\theta) \equiv \exp[\theta(a\tilde{a} - a^{+}\tilde{a}^{+})], \tag{4}$$

is the thermal squeezing operator which transforms the zero-temperature vacuum state $|0, \tilde{0}\rangle$ into the thermal vacuum state $|0(\beta)\rangle$, and $\tanh\theta = e^{-\hbar\omega/(2kT)} = e^{-\beta\hbar\omega/2}$ ($\beta = \frac{1}{kT}$, k is the Boltzmann constant).

Louisell quantized a L-C circuit as a harmonic oscillator and has calculated its zero-point quantum fluctuation. At finite temperature the vacuum state of the L-C circuit becomes the thermal vacuum state. In Ref. (Fan *et al.*, 1998), we have introduced the coherent thermal state representation

$$|\tau\rangle = \exp\left[-\frac{1}{2}|\tau|^2 + \tau a^{+} - \tau^{*}\tilde{a}^{+} + a^{+}\tilde{a}^{+}\right]|0, \tilde{0}\rangle. \tag{5}$$

The complete set and orthogonality of the state $|\tau\rangle$ are, respectively,

$$\int \frac{d^2\tau}{\pi}\, |\tau\rangle\, \langle\tau| = 1,\ \langle\tau'|\tau\rangle = \pi\delta(\tau - \tau')\delta(\tau^{*} - \tau'^{*}). \tag{6}$$

By using the state $|\tau\rangle$, the thermal squeezing operator is neatly expressed as

$$S(\theta) = \int \frac{d^2\tau}{\pi\mu} \, |\tau/\mu\rangle \langle\tau| , \, \mu^2 = \frac{1 + \tanh\theta}{1 - \tanh\theta}, \tag{7}$$

thus

$$S(\theta)\,|\tau\rangle = 1/\mu \, |\tau/\mu\rangle \,, \tag{8}$$

$$|0(\beta)\rangle = \int \frac{d^2\tau}{\pi\mu} \, |\tau/\mu\rangle \langle\tau|\, 0, \tilde{0}\rangle = \int \frac{d^2\tau}{\pi\mu} \, |\tau/\mu\rangle \, e^{-\frac{1}{2}|\tau|^2}. \tag{9}$$

Now the vacuum state of the L-C circuit is in $|0(\beta)\rangle$.

3. THE THERMAL WIGNER OPERATOR

For a density operator ρ the Wigner function is defined as (Fan, 2002)

$$W(p, q) = \mathrm{Tr}(\rho\Delta(p, q)), \tag{10}$$

where $\Delta(p, q)$ is the Wigner operator, usually in terms of the coordinate eigenvector is expressed as

$$\Delta(p, q) = \int_{-\infty}^{\infty} dv \left| q - \frac{v}{2} \right\rangle \left\langle q + \frac{v}{2} \right| e^{-ipv}, \tag{11}$$

and its normally ordered form is

$$\Delta(p, q) = \frac{1}{\pi} : e^{-(q-Q)^2 - (p-P)^2} : . \tag{12}$$

$W(p, q)$ in phase space has its physical meaning as marginal probability distribution in coordinate space and momentum space, respectively. This can be clearly seen from the marginal integration of the normally ordered form of Wigner operator (Fan, 2005)

$$\int dp\Delta(p, q) = \frac{1}{\pi} \int dp : e^{-(q-Q)^2-(p-P)^2} := \frac{1}{\sqrt{\pi}} : e^{-(q-Q)^2} := |q\rangle \langle q| \,, \tag{13}$$

$$\int dq\Delta(p, q) = \frac{1}{\sqrt{\pi}} : e^{-(p-P)^2} := |p\rangle \langle p| \,. \tag{14}$$

Recall that the Wigner operator is the integration kernel of Weyl correspondence (Fan, 2004)

$$A = \int dpdq A(p, q)\Delta(p, q), \tag{15}$$

so the expectation value of operator A in the thermal vacuum state is

$$\langle 0(\beta)|\, A\, |0(\beta)\rangle = \int dpdq A(p,q)\, \langle 0(\beta)|\, \Delta(p,q)\, |0(\beta)\rangle\,, \tag{16}$$

where

$$\langle 0(\beta)|\, \Delta(p,q)\, |0(\beta)\rangle = W_T(p,q) \tag{17}$$

is named the Wigner function of thermal vacuum state, the subscript "T" means "thermal". Thus a mixed state's Wigner function can be equivalent to $\langle 0(\beta)|\, \Delta(p,q)\, |0(\beta)\rangle$. In Ref. (Fan, 2005) by virtue of the coherent thermal state representation we introduce the thermal Wigner operator

$$\Delta_T(\sigma,\gamma) = \int \frac{d^2\tau}{\pi}\, |\sigma-\tau\rangle\, \langle\sigma+\tau| \exp(\tau\gamma^* - \gamma\tau^*), \tag{18}$$

where $\gamma = \alpha + \varepsilon^*, \sigma = \alpha - \varepsilon^*, \alpha = \frac{1}{\sqrt{2}}(q + ip)$. By using the technique of integration within an ordered product (IWOP) of operators (Fan *et al.*, 1987), from Eq. (17), we can obtain

$$\Delta_T(\sigma,\gamma) = \pi^{-2} : \exp[-2(a^+ - \alpha^*)(a-\alpha) - 2(\tilde{a}^+ - \varepsilon^*)(\tilde{a} - \varepsilon)] :, \tag{19}$$

its relationship with $\Delta(p,q)$ is

$$2\mathrm{Tr}\Delta_T(\sigma,\gamma) = \Delta(p,q) = 2\int d^2\varepsilon\, \Delta_T(\sigma,\gamma). \tag{20}$$

4. WIGNER FUNCTION OF THERMAL VACUUM STATE

From Eqs. (6), (7) and (18), we deduce

$$S^{-1}(\theta)\Delta_T(\sigma,\gamma)S(\theta) = \Delta_T(\gamma/\mu.\mu\sigma), \quad \tanh\theta = \frac{\mu^2-1}{\mu^2+1}. \tag{21}$$

By virtue of Eq. (21), we can calculate the Wigner function of thermal vacuum state

$$\begin{aligned} W_T(p,q) &= \langle 0(\beta)|\, \Delta(p,q)\, |0(\beta)\rangle = 2\int d^2\varepsilon \langle 0,\tilde{0}|S^{-1}(\theta)\Delta_T(\sigma,\gamma)S(\theta)|0,\tilde{0}\rangle \\ &= 2\int \frac{d^2\varepsilon}{\pi^2} \exp\left(-\,|\sigma|^2\,\mu^2 - \frac{|\gamma|^2}{\mu^2}\right) \\ &= 2\int \frac{d^2\varepsilon}{\pi^2} \exp\left[(\varepsilon\alpha + \varepsilon^*\alpha^*)\left(\mu^2 - \frac{1}{\mu^2}\right)\right. \end{aligned}$$

$$- |\varepsilon|^2 \left(\mu^2 + \frac{1}{\mu^2}\right) - \frac{\mu^4+1}{\mu^2} |\alpha|^2 \Bigg]$$

$$= \frac{2\mu^2}{\pi(\mu^4+1)} \exp\left(\frac{-4\mu^2}{\mu^4+1} |\alpha|^2\right)$$

$$= \frac{2\mu^2}{\pi(\mu^4+1)} \exp\left(\frac{-2\mu^2}{\mu^4+1}(q^2+p^2)\right), \tag{22}$$

due to

$$\frac{2\mu^2}{\mu^4+1} = \frac{1-e^{-\beta\,\hbar\omega}}{1+e^{-\beta\,\hbar\omega}}, \quad \tanh\theta = \exp\left(-\frac{\hbar\omega}{2kT}\right), \tag{23}$$

thus

$$W_T(p,q) = \frac{1-e^{-\beta\,\hbar\omega}}{\pi(1+e^{-\beta\,\hbar\omega})} \exp\left(-\frac{1-e^{-\beta\,\hbar\omega}}{1+e^{-\beta\,\hbar\omega}}(q^2+p^2)\right). \tag{24}$$

5. MARGINAL DISTRIBUTION OF THE WIGNER FUNCTION OF THE THERMAL VACUUM STATE FOR L-C CIRCUIT

Marginal distribution of the Wigner function of the thermal vacuum state for L-C circuit in the coordinate space and momentum space are, respectively,

$$\int dq W_T(p,q) = \frac{1}{\sqrt{\pi}}\sqrt{\frac{1-e^{-\beta\,\hbar\omega}}{1+e^{-\beta\,\hbar\omega}}} \exp\left[-\frac{1-e^{-\beta\,\hbar\omega}}{1+e^{-\beta\hbar\omega}} p^2\right]$$

$$= \frac{1}{\sqrt{\pi}} \tanh^{1/2}(\beta\hbar\omega/2) \exp[-p^2 \tanh(\beta\ \hbar\omega/2)], \tag{25}$$

$$\int dp W_T(p,q) = \frac{1}{\sqrt{\pi}}\sqrt{\frac{1-e^{-\beta\,\hbar\omega}}{1+e^{-\beta\,\hbar\omega}}} \exp\left[-\frac{1-e^{-\beta\,\hbar\omega}}{1+e^{-\beta\hbar\omega}} q^2\right]$$

$$= \frac{1}{\sqrt{\pi}} \tanh^{1/2}(\beta\ \hbar\omega/2) \exp[-q^2 \tanh(\beta\ \hbar\omega/2)], \tag{26}$$

where

$$\tanh\theta = e^{-\hbar\omega/(2kT)} = e^{-\beta\hbar\omega/2}. \tag{27}$$

The Weyl correspondence of Q^2 is

$$Q^2 \to \frac{\hbar(\alpha+\alpha^*)^2}{2\omega L}, \quad P^2 \to -\frac{L\omega\hbar(\alpha-\alpha^*)^2}{2}, \tag{28}$$

thus the temperature-related energy stored in the capacitance is

$$\begin{aligned} H_C &= \left\langle 0(\beta) \left| \frac{Q^2}{2C} \right| 0(\beta) \right\rangle = \frac{1}{2C} \int dpdq \frac{\hbar(\alpha+\alpha^*)^2}{2\omega L} \langle 0(\beta)| \Delta(p,q) |0(\beta)\rangle \\ &= \frac{1}{2C} \int dq \frac{\hbar(\alpha+\alpha^*)^2}{2\omega L} \int dp W_T(p,q) \\ &= \frac{\hbar}{2\sqrt{\pi}\omega LC} \tanh^{1/2}(\beta\hbar\omega/2) \int dq q^2 \exp[-q^2 \tanh(\beta\hbar\omega/2)] \\ &= \frac{\hbar\omega}{4} \coth \frac{\beta\ \hbar\omega}{2}, \end{aligned} \tag{29}$$

the temperature-related energy stored in the inductance is

$$\begin{aligned} H_L &= \left\langle 0(\beta) \left| \frac{P^2}{2L} \right| 0(\beta) \right\rangle = -\frac{1}{2L} \int dpdq \frac{L\omega\ \hbar(\alpha-\alpha^*)^2}{2} \langle 0(\beta)| \Delta(p,q) |0(\beta)\rangle \\ &= -\frac{1}{2L} \int dp \frac{L\omega\ \hbar(\alpha-\alpha^*)^2}{2} \int dq W_T(p,q) \\ &= \frac{\hbar\omega}{2\sqrt{\pi}} \tanh^{1/2}(\beta\ \hbar\omega/2) \int dp p^2 \exp[-p^2 \tanh(\beta\ \hbar\omega/2)] \\ &= \frac{\hbar\omega}{4} \coth \frac{\beta\ \hbar\omega}{2}, \end{aligned} \tag{30}$$

and the energy of the system is

$$H = H_C + H_L = \frac{\hbar\omega}{2} \coth \frac{\beta\ \hbar\omega}{2}, \tag{31}$$

which is just the temperature-related zero-energy of the system.

Here we endow the Wigner function with a new physical meaning, i.e., its marginal distributions' statistical average for $q^2/(2C)$ and $p^2/(2L)$ are the temperature-related energy stored in the capacitance and in the inductance of the mesoscopic L-C circuit at finite temperature, respectively. This new view endows the Wigner function with a more realistic physical interpretation.

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REFERENCES

Fan, H. Y., Zaidi, H. R., and Klauder, J. R. (1987). *Physical Review* D **35**, 1831.
Fan, H. Y. and Fan, Y. (1998). *Physics Letters* A **246**, 242.
Fan, H. Y. and Liang, X. T. (2000). *Chinese Physics Letters* **17**, 174.
Fan, H. Y. (2002). *Communications in Theoretical Physics* **38**, 533.
Fan, H. Y. (2004). *Entangled State Representations in Quantum Mechanics and Their Applications* [M]. Shanghai: Jiao Tong University Press, 13 (in Chinese).
Fan, H. Y. (2005). *From Quantum Mechanics to Quantum Optics. Development of the Mathematical Physics* [M]. Shanghai: Jiao Tong University Press, 107, 207 (in Chinese).
Louisell, W. H. (1973). *Quantum Statistical Properties of Radiation* [M]. New York: John Wiley. Chapter 4.
Song, T. Q. (2003). *International Journal of Theoretical Physics* **42**, 793.
Takahashi, Y. and Umezawa, H. (1975). Thermo field dynamics. *Collective Phenomena* **2**, 55.
Wang, J. S., Liu, T. K., and Zhan, M. S. (2000). *International Journal of Theoretical Physics* **39**, 2013.